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ABSTRACT 18455

A general perturbation theory for nonlinear oscillations in a collisionless plasma is developed which is not restricted by any assumptions regarding the randomness of the phases. Summation of the secular terms of the perturbation theory leads to equations for "slow" processes. In the case of sufficiently broad wave packets, these equations change into the familiar equations of the quasilinear theory for a weakly turbulent plasma. The converse limiting case -- the development of a periodic wave in the quasilinear approximation -- is investigated in detail. Author

1. INTRODUCTION

Nonlinear oscillations in a collisionless plasma, which are described /515* by the equations:[(1)]

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* Note: Numbers in the margin indicate pagination in the original foreign text.

(1) For purposes of simplicity, we shall investigate potential oscillations without a magnetic field, although all the results can be extended to the general case without any great difficulty.

$$\frac{\partial F_j}{\partial t} + \mathbf{v}\frac{\partial F_j}{\partial \mathbf{r}} + \frac{e_j}{m_j}\mathbf{E}\frac{\partial F_j}{\partial \mathbf{v}} = 0 \tag{1.1}$$

$$\nabla\mathbf{E} = 4\pi \sum_j e_j N_j \int F_j d\mathbf{v} \tag{1.2}$$

(j - index indicating the type of particles), have been investigated in several studies [see the abstract (Ref. 1, 2), where a detailed bibliography is presented and also more recent articles (Ref. 3-10)]. One distinguishing feature of the methods developed in these studies is the utilization of the so-called approximation of "random phases", so that their results are only applicable to a turbulent plasma, where the width of the wave packet is quite large. In several cases (for example, in a finite plasma) the problem arises of studying the dynamics of nonlinear periodic waves which are characterized by the discrete selection of wave numbers k. Naturally, in this case the approximation of random phases is not applicable.

This article examines the general perturbation theory for plasma oscillations, which is not limited to assumptions regarding the random nature of the phases. A formal expansion is carried out in powers of the oscillation field; then the orders of secular terms are isolated and summed in general perturbation theory series, similarly to the work of Van Hove (Ref. 11), Prigogine (Ref. 12), and Balescu (Ref. 13) in obtaining kinetic equations for slightly nonideal systems. Summation of the "main" orders of secular terms leads to quasilinear equations which describe the reverse effect of the oscillations on the distribution function of plasma particles. The applicability of these equations, however, is not confined to conditions regarding the wave packet width. If this width is large enough, these equations change into the well-known equations of the quasilinear theory for a slightly turbulent plasma (Ref. 14, 15). In the opposite, limiting case, equations are obtained for a "monochromatic" wave. The solution of these equations, which is given in this article, describes the development of the plasma distribution function and the field of the "monochromatic" wave, with allowance for the reverse effect of the wave and the distribution function. /516

2. PERTURBATION THEORY. SUMMATION OF THE TERMS

Following the work of Landau (Ref. 16), we shall look for a solution for equations (1.1) and (1.2) with the initial condition[2]

$$F(0, \mathbf{r}, \mathbf{v}) = \sum_{\mathbf{k}*} F^0_{\mathbf{k}}(\mathbf{v}) e^{i\mathbf{k}\mathbf{r}} = f(\mathbf{v}) + \sum_{\mathbf{k}\neq 0} g_{\mathbf{k}}(\mathbf{v}) e^{i\mathbf{k}\mathbf{r}} \tag{2.1}$$

(For purposes of brevity, the index indicating the type of particles is omitted from this point on.) Expanding $F(t,\mathbf{r},\mathbf{v})$ and $\mathbf{E}(t,\mathbf{r})$ in Fourier series:

$$F(t, \mathbf{r}, \mathbf{v}) = \sum_{\mathbf{k}} F_{\mathbf{k}}(t, \mathbf{v}) e^{i\mathbf{k}\mathbf{r}}, \quad \mathbf{E}(t, \mathbf{r}) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}} \tag{2,2}$$

and applying the Laplace transformation to terms dependent on time[3]:

(2) The normalized volume is assumed to equal unity.

(3) The corresponding terms in the Laplace representation are analytic in the upper halfplane ω, except possibly in a certain vicinity of the actual axis. The inverse transformations have the form

$$\mathbf{E}_{\mathbf{k}}(t) = \frac{1}{2\pi} \int\limits_{-\infty+i\delta}^{\infty+i\delta} \mathbf{E}_{\mathbf{k}}(\omega) e^{-i\omega t} d\omega, \qquad \delta > 0.$$

It is also advantageous to keep in mind the relationship of the convolution:

$$\int\limits_0^\infty F_1(t)F_2(t)e^{i\omega t}\, dt = \frac{1}{2\pi}\int d\omega' F_1(\omega')F_2(\omega-\omega') = \frac{1}{(2\pi)^2 i}\int \frac{d\omega' d\omega''}{\omega'+\omega''-\omega} F_1(\omega')F_2(\omega''),$$

where integration is carried out along lines lying in the upper halfplane, satisfying the condition Im ω > Im ω' + Im ω''.

* This will be designated by K.

$$F_{\mathbf{k}}(\omega, \mathbf{v}) = \int_0^\infty F_{\mathbf{k}}(t, \mathbf{v}) e^{i\omega t}\, dt, \quad \mathbf{E}_{\mathbf{k}}(\omega) = \int_0^\infty \mathbf{E}_{\mathbf{k}}(t) e^{i\omega t}\, dt, \tag{2.3}$$

We obtain the following integral equation instead of (1.1):

$$F_{\mathbf{k}}(\omega, \mathbf{v}) = \frac{1}{\omega - \mathbf{kv}} \frac{e}{im} \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \int \frac{d\omega'}{2\pi} \mathbf{E}_{\mathbf{k}'}(\omega') \frac{\partial F_{\mathbf{k}''}(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}} + \frac{iF_{\mathbf{k}}^0(\mathbf{v})}{\omega - \mathbf{kv}}, \tag{2.4}$$

where $F_K^0(\mathbf{v})$ is determined by the initial conditions (2.1). In obtaining (2.4), we employed the relationship of convolution [see footnote[3]].

Let us expand the solution of equation (2.4) in power series of the field $\mathbf{E}$:

$$F_{\mathbf{k}}(\omega, \mathbf{v}) = \sum_{n=0}^{\infty} F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}), \tag{2.5}$$

where $F_K^{(n)}(\omega,\mathbf{v}) \sim E^n$, $\mathbf{E}_k \sim g_k(\mathbf{v})$. We employ $f(\mathbf{v})$ for the zero approximation [see (2.1)]. The first approximation is given by the relationship /517 ($k \neq 0$)

$$F_{\mathbf{k}}^{(1)}(\omega, \mathbf{v}) = \frac{e}{im} \frac{\mathbf{E}_{\mathbf{k}}(\omega)}{\omega - \mathbf{kv}} \frac{df}{d\mathbf{v}} + \frac{ig_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{kv}}. \tag{2.6}$$

The following recurrence formula for $F_K^{(n)}(\omega,\mathbf{v})$ in the case of $n \geqslant 2$ follows from (2.4):

$$F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}) = \frac{1}{\omega - \mathbf{kv}} \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \frac{e}{im} \int \frac{d\omega'}{2\pi} \mathbf{E}_{\mathbf{k}'}(\omega') \frac{\partial F_{\mathbf{k}''}^{(n-1)}(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}}. \tag{2.7}$$

By means of (2.7), (2.6), we obtain the expression for the general term of series (2.5):

$$\begin{aligned} F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}) = \left(\frac{e}{2\pi i m}\right)^n \sum_{\mathbf{k}=\mathbf{k}_1+\ldots+\mathbf{k}_n} \Bigg\{ \int d\omega_1\, d\omega_2 \ldots d\omega_n \frac{\mathbf{E}_{\mathbf{k}_1}(\omega_1)}{\omega - \mathbf{kv}} \times \\ \times \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{\mathbf{k}_2}(\omega_2)}{\omega - \mathbf{kv} - \omega_1 + \mathbf{k}_1\mathbf{v}} \frac{\partial}{\partial \mathbf{v}} \cdots \frac{E_{\mathbf{k}_n}(\omega_n)}{\omega - \mathbf{kv} - \sum_{i=1}^{n-1} (\omega_i - \mathbf{k}_i\mathbf{v})} \frac{\partial}{\partial \mathbf{v}} \times \\ \times \frac{f(\mathbf{v})}{\omega - \mathbf{kv} - \sum_{i=1}^{n} (\omega_i - \mathbf{k}_i\mathbf{v})} - \frac{2\pi m}{e} \int d\omega_1 d\omega_2 \ldots d\omega_n \frac{\mathbf{E}_{\mathbf{k}_1}(\omega_1)}{\omega - \mathbf{kv}} \times \end{aligned} \tag{2.8}$$

$$\times \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{\mathbf{k}_2}(\omega_2)}{\omega - \mathbf{kv} - \omega_1 + \mathbf{k}_1\mathbf{v}} \frac{\partial}{\partial \mathbf{v}} \cdots \frac{\mathbf{E}_{\mathbf{k}_{n-1}}(\omega_{n-1})}{\omega - \mathbf{kv} - \sum_{i=1}^{n-2} (\omega_i - \mathbf{k}_i\mathbf{v})} \times$$

$$\left. \times \frac{\partial}{\partial \mathbf{v}} \frac{g_{\mathbf{k}_n}(\mathbf{v})}{\omega - \mathbf{kv} - \sum_{i=1}^{n-1} (\omega_i - \mathbf{k}_i\mathbf{v})} \right\}. \qquad (2.8)\ \text{(cont.)}$$

It is possible to provide a simple, graphic representation of the general term, by introducing its diagrammatic representation (Figure 1). $\mathbf{E}_{K_S}(\omega_S)$ correspond to the solid vertical lines in Figure 1. The horizontal lines designate the "lines of propagation"

$$\left[\omega - \mathbf{kv} - \sum_{i=1} (\omega_i - \mathbf{k}_i\mathbf{v}) \right]^{-1} \quad (0 \leqslant s \leqslant n)\text{:}$$

The integrodifferential operator corresponds to the s-apex

$$\frac{e}{im} \frac{\partial}{\partial \mathbf{v}} \int \frac{d\omega_s}{2\pi};$$

The circle at the right term in Figure 1 expresses the function $f(\mathbf{v})$, and the vertical dashed line at the second term - the function $ig_K(\mathbf{v})$. The order of the diagram is determined by the number of vertical lines (including the dashed line). It can be seen from Figure 1 that the term of each order of the perturbation theory in the distribution function consists of two individual parts: The parts depending on $f(\mathbf{v})$ (with the circle at the right end of Figure 1), and the parts dependent on $g_K(\mathbf{v})$ (with one dashed line at the right end of the diagram). /518

Let us now examine equation (1.2) for the wave field. Taking into account (2.2), (2.3), (2.5), we can rewrite (1.2) in the following form

$$\mathbf{kE}_{\mathbf{k}}(\omega) = -4\pi ieN \sum_{n=1}^{\infty} \int F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v})\, d\mathbf{v} \qquad (2.9)$$

(in order to abbreviate the formulas, we shall omit summation symbols corresponding to types of particles from this point on; reduction of the

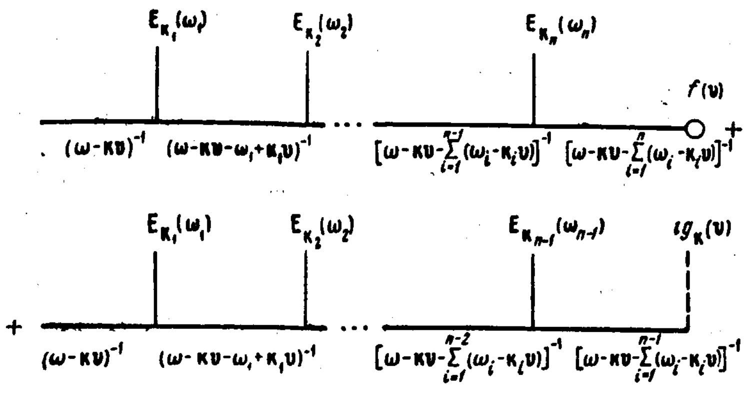

Figure 1

corresponding summation in the final formulas entails no difficulties). Substituting $F_K^{(1)}(\omega,\mathbf{v})$ from (2.6) in the first approximation term in the right part of (2.9) and combining the terms which are linear in E, we can rewrite (2.9) in the following form

$$\varepsilon_{\mathbf{k}}(\omega)\mathbf{E}_{\mathbf{k}}(\omega)=\frac{4\pi eN}{k^2}\mathbf{k}\int\frac{g_{\mathbf{k}}(\mathbf{v})\,d\mathbf{v}}{\omega-\mathbf{kv}}-\frac{4\pi ieN}{k^2}\mathbf{k}\sum_{n=2}^{\infty}\int F_{\mathbf{k}}^{(n)}(\omega,\mathbf{v})\,d\mathbf{v}, \tag{2.10}$$

where $\varepsilon_K(\omega)$ is the dielectric constant of the plasma:

$$\varepsilon_{\mathbf{k}}(\omega)=1+\frac{\omega_0^2}{k^2}\int\frac{d\mathbf{v}}{\omega-\mathbf{kv}}\mathbf{k}\frac{df}{d\mathbf{v}},\qquad \omega_0^2=\frac{4\pi e^2N}{m}. \tag{2.11}$$

The second term in the right part of (2.10) describes nonlinear effects. If we disregard it, the well-known equation for the operation field in the linear approximation obtained by Landau (Ref. 16) is obtained

$$\mathbf{E}_{\mathbf{k}}(t)=\int\frac{\mathbf{r}_{\mathbf{k}}(\omega)}{\varepsilon_{\mathbf{k}}(\omega)}e^{-i\omega t}\frac{d\omega}{2\pi}\approx\mathbf{E}_{\mathbf{k}}^0e^{-i\omega_{\mathbf{k}}t}; \tag{2.12}$$

$$\mathbf{r}_{\mathbf{k}}(\omega)=\frac{4\pi eN}{k^2}\mathbf{k}\int\frac{g_{\mathbf{k}}(\mathbf{v})\,d\mathbf{v}}{\omega-\mathbf{kv}},\qquad \mathbf{E}_{\mathbf{k}}^0=\frac{\mathbf{r}_{\mathbf{k}}(\omega_{\mathbf{k}})}{\varepsilon_{\mathbf{k}}'(\omega_{\mathbf{k}})},\qquad \varepsilon_{\mathbf{k}}'(\omega_{\mathbf{k}})=\frac{d\varepsilon_{\mathbf{k}}(\omega)}{d\omega}\Big|_{\omega=\omega_{\mathbf{k}}}; \tag{2.13}$$

$$\omega_{\mathbf{k}}=\omega_{\mathbf{k}}^0+i\gamma_{\mathbf{k}},\qquad \gamma_{\mathbf{k}}=\pi\frac{\omega_0^2}{\varepsilon_{\mathbf{k}}'k^2}\frac{df}{dv}\Big|_{v=\omega_{\mathbf{k}}^0/k} \tag{2.14}$$

where $\mathbf{E}_K^0$ is the oscillation amplitude which, as was indicated above, has the same order of magnitude as the initial perturbation $g_K(\mathbf{v})$, and ω_K is the

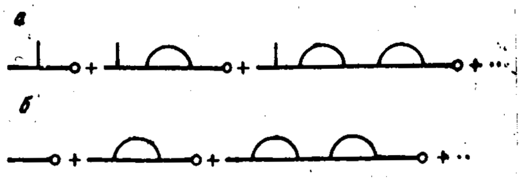

Figure 2

complex oscillation frequency which is the root of the dispersion equation $\varepsilon_K(\omega) = 0$ with the largest imaginary part. Terms corresponding to /519 other roots of the dispersion equation are omitted in the right part of (2.2); these terms are exponentially small for $t \to \infty$. We shall assume that the condition

$$\gamma_{\mathbf{k}} / \omega_{\mathbf{k}} \ll 1, \tag{2.15}$$

is fulfilled, without which the results presented below are inapplicable.

In the formal expansion of (2.5), not all of the nonlinear terms are actually small (in the sense that they can become large for rather large t after conversion to the t-representation). In order to obtain the correct asymptotic behavior for large t, such terms must be isolated and summed. Instead of the exact field components $\mathbf{E}_{K_S}(\omega_S)$ which satisfy equation (2.10), let us first substitute their values in the linear approximation (2.12) in the expression for the general term (2.8), in order to clarify the "large" terms. Let us disregard the imaginary parts of the frequencies ω_{K_S}. In the Laplace representation, the corresponding expressions will have the following form:

$$\mathbf{E}_{\mathbf{k}_s}(\omega_s) = \frac{i\mathbf{E}^0_{\mathbf{k}_s}}{\omega_s - \omega^0_{k_s}}, \tag{2.16}$$

where $\mathbf{E}^0_{K_S}$ is the oscillation amplitude in the linear approximation determined by equation (2.13). Due to the simple form of $\mathbf{E}_{K_S}(\omega_S)$ in (2.16), we can readily carry out integration over all ω_S in the general term (2.8), as a result of which the quantities ω_s are replaced by $\omega_{k_s}^{\,0}$, and $\mathbf{E}_{K_s}(\omega_s)$ is replaced by $\mathbf{E}_{K_s}^{\,0}$.

We should now note that among the different diagrams in the sum with respect to k_s in Figure 1, there will be diagrams in which the series of lines corresponds to the conjugate field components ($K_s = -K_{s+1}$, $\omega_{K_s}^0 = -\omega_{K_{s+1}}^0$). We shall call these lines coupled lines. The coupled lines in the diagrams are closed in the form of loops (see Figure 2). The propagators on both sides of the loop are the same, which leads to the appearance of multiple poles with respect to ω in $F_K^{(n)}(\omega,\mathbf{v})$. In the t-representation, the corresponding terms will be secular, i.e., they will be proportional to t^r, where r+1 is the multiplicity of the pole in the Laplace representation.

By way of a typical example, let us examine the term expressed by the diagram of the second order in Figure 2,b. After substituting the approximation (2.16) for the field $\mathbf{E}_{K_s}(\omega_s)$ and performing integration over ω_s, we obtain for it

$$\left(\frac{e}{im}\right)^2 \sum_{\mathbf{q}} \frac{\mathbf{E}_{\mathbf{q}}^0}{\omega} \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{-\mathbf{q}}^0}{\omega - \omega_{\mathbf{q}}^0 + \mathbf{q}\mathbf{v}} \frac{\partial}{\partial \mathbf{v}} \frac{f(\mathbf{v})}{\omega} . \tag{2.17}$$

This expression, which is one of the contributions to the Fourier component of the distribution function with k = 0, has a second-order pole for $\omega = 0$; correspondingly, it provides a secular term which is proportional to t in the t-representation. Similarly, for the term expressed by the third-order diagram in Figure 2,a (which makes a contribution to the k-component /520 of the Fourier distribution function) we obtain

$$\left(\frac{e}{im}\right)^3 \sum_{\mathbf{q}} \frac{\mathbf{E}_k^0}{\omega - \mathbf{k}\mathbf{v}} \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{\mathbf{q}}^0}{\omega - \omega_k^0} \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{-\mathbf{q}}^0}{\omega - \omega_k^0 - \omega_{\mathbf{q}}^0 + \mathbf{q}\mathbf{v}} \frac{\partial}{\partial \mathbf{v}} \frac{f(\mathbf{v})}{\omega - \omega_k^0} . \tag{2.18}$$

Expression (2.18) has a second-order pole for $\omega = \omega_K^{\,0}$ (in the t-representation, it is proportional to $t \exp(-i\omega_K{}^0 t)$.

It was assumed above that (2.16) was substituted, instead of $\mathbf{E}_K(\omega)$. If we substitute exact $\mathbf{E}_K(\omega)$ or (2.16), but with the complex frequency $\omega_K = \omega_K{}^0 + i\gamma_K$, then secular terms will not appear after conversion to the t-representation. However, if $|\mathbf{E}_K(t)|$ slowly depends on time, then terms containing coupled $\mathbf{E}_K$ will not be secular, but will be large for a sufficiently large t [4].

Thus, for sufficiently large times the nonlinear terms in the equation for the field (2.9) will not only lead to small corrections to the solution of the corresponding linearized equation, but can also completely change the solution. In order to obtain the correct asymptotic behavior of the field for large t, the "large" terms indicated above must be summed. Generally speaking, the term of the nth order (2.9) contains secular components which have a degree of secularness (i.e., the power of t which is contained in this component after conversion to the t-representation). Terms having the maximum degree of secularness for a given order n will be called main terms. We shall confine ourselves in this article to summing only the main secular terms [terms having a small degree of secularness were investigated in (Ref. 18)]. It can be shown (Ref. 18) that the main secular terms in the right part of equation (2.9) are expressed by the diagrams drawn in

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(4) If we substitute (2.16) with complex ω_K, instead of $\mathbf{E}_K(\omega)$, then - as can be readily corroborated - we will have $(\omega_K^0/\gamma_K)^r \gg 1$ instead of the secular factor t^r, so that the corresponding terms must be assumed to be large, as was done previously. The main difference between the results of our study and the results obtained by Montgomery (Ref. 17) lies in the fact that the indicated large terms are not summed in the perturbation theory developed in (Ref. 17).

Figure 2,a. Let us designate the sum of the diagrams in Figure 2,a by $\Phi_K(\omega,\mathbf{v})$. It can be readily shown that this quantity is simply expressed by the function $\Phi(\omega,\mathbf{v})$ which represents the sum of all the diagrams shown in Figure 2,b, namely,

$$\Phi_{\mathbf{k}}(\omega,\mathbf{v}) = \frac{e}{im}\int \frac{d\omega'}{2\pi}\frac{\mathbf{E}_{\mathbf{k}}(\omega')}{\omega-\mathbf{kv}}\frac{\partial\Phi(\omega-\omega',\mathbf{v})}{\partial\mathbf{v}}. \tag{2.19}$$

With allowance for only the main secular terms, equation (2.9) assumes only the following form

$$\begin{aligned}\mathbf{k}\mathbf{E}_{\mathbf{k}}(\omega) &= 4\pi eN\int \frac{g_{\mathbf{k}}(\mathbf{v})}{\omega-\mathbf{kv}}d\mathbf{v} - 4\pi ieN\int \Phi_{\mathbf{k}}(\omega,\mathbf{v})\,d\mathbf{v} = \\ &= 4\pi eN\int \frac{g_{\mathbf{k}}(\mathbf{v})}{\omega-\mathbf{kv}}d\mathbf{v} - \omega_0^2\int\frac{d\omega'}{2\pi}\int\frac{d\mathbf{v}E_{\mathbf{k}}(\omega')}{\omega-\mathbf{kv}}\frac{\partial\Phi(\omega-\omega',\mathbf{v})}{\partial\mathbf{v}}.\end{aligned} \tag{2.20}$$

Thus we obtain the main equation for the field in the form

$$\int\frac{d\omega'}{2\pi}\varepsilon_{\mathbf{k}}(\omega,\omega')\mathbf{E}_{\mathbf{k}}(\omega') = 4\pi eN\frac{\mathbf{k}}{k^2}\int\frac{g_{\mathbf{k}}(\mathbf{v})}{\omega-\mathbf{kv}}d\mathbf{v}, \tag{2.21}$$

$$\varepsilon_{\mathbf{k}}(\omega,\omega') = \frac{i}{\omega-\omega'} + \frac{\omega_0^2}{k^2}\int\frac{d\mathbf{v}}{\omega-\mathbf{kv}}\mathbf{k}\frac{\partial\Phi(\omega-\omega',\mathbf{v})}{\partial\mathbf{v}}. \tag{2.22}$$

Equation (2.21) differs from the linear equation for the field in the /521
fact that, instead of the dielectric coefficient $\varepsilon_K(\omega)$, it contains the integral operator with the series $\varepsilon_K(\omega,\omega')$ which is expressed by the function $\Phi(\omega,\mathbf{v})$. If we replace $\Phi(\omega,\mathbf{v})$ by the zero approximation of the distribution function (the first component in Figure 2,b) - i.e., if we set $\Phi(\omega,\mathbf{v}) = if(\mathbf{v})/\omega$ - then (2.21) changes into the linear equation for the field.

In order to obtain the total system of equations describing the development of the oscillations, we must obtain the equation for the function $\Phi(\omega,\mathbf{v})$. This quantity represents a distribution function averaged over spatial pulsations. From this point on, we shall call it the distribution function of the background. It follows from the form of the diagrams in Figure 2,b that $\Phi(\omega,\mathbf{v})$ satisfies the equation

$$-i\omega\Phi(\omega, \mathbf{v}) = f(\mathbf{v}) + \\ + \frac{ie^2}{m^2}\sum_{\mathbf{q}}\int \mathbf{E}_{-\mathbf{q}}(\omega')\frac{\partial}{\partial \mathbf{v}}\frac{\mathbf{E}_{\mathbf{q}}(\omega'')}{\omega - \mathbf{q}\mathbf{v} - \omega'}\frac{\partial\Phi(\omega - \omega' - \omega'', \mathbf{v})}{\partial \mathbf{v}}\frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}, \tag{2.23}$$

which is an analog of the Dyson equation in the quantum field theory.

We shall call equations (2.21) and (2.23) generalized quasilinearized equations. [5] Under certain additional assumptions, they change to the well-known equations of the quasilinear theory for a weakly turbulent plasma (Ref. 14, 15). As will be seen later on, the conditions under which this occurs stipulate that the width of the plasma oscillation spectrum must be large enough. In another limiting case, when the spectrum is very narrow - for example, when "a monochromatic" (more accurately, a periodic) wave is excited in a plasma - equation (2.23) has completely different properties and solutions, while retaining the same meaning as for a weakly turbulent plasma (with allowance for the reverse effect of the wave on the distribution function).

Let us now examine in greater detail the manner in which equation (2.23) changes into a quasilinear equation for a weakly turbulent plasma. Let us first substitute $\mathbf{E}_{\mathbf{q}}(\omega)$ in the right part in the form of (2.16) (this means that we have disregarded the dependence of the field amplitude $|\mathbf{E}_{\mathbf{q}}(t)| = \mathbf{E}_{\mathbf{q}}^{0}$ on time), and let us perform integration over ω' and ω''. As a result, we obtain

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(5) We must point out that in summing the diagrams leading to equations (2.21) and (2.23), we did not employ approximation (2.16) for the field. We only needed the latter in order to clarify "large" terms.

$$-i\omega\Phi(\omega, \mathbf{v}) = f(\mathbf{v}) + i\left(\frac{e}{m}\right)^2 \sum_{\mathbf{q}} \frac{|\mathbf{E}_{\mathbf{q}}{}^0|^2}{\mathbf{q}^2} \mathbf{q} \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega + \omega_{\mathbf{q}} - \mathbf{q}\mathbf{v}} \mathbf{q} \frac{\partial \Phi(\omega, \mathbf{v})}{\partial \mathbf{v}}. \quad (2.24)$$

The region of the ω values, where $\Phi(\omega,\mathbf{v})$ is large, is related to the characteristic time τ required by the function $\Phi(t,\mathbf{v})$ to change by the relationship $|\omega| \sim \tau^{-1}$. Let us assume that the following condition is fulfilled

$$\tau^{-1} \ll \overline{|\omega_{\mathbf{q}}{}^0 - \mathbf{q}\mathbf{v}|}, \quad (2.25)$$

where averaging is carried out over $\mathbf{q}$. We can then disregard ω in the denominator of the right part of (2.24). However, we must set $\omega \to i0$ because ω lies in the upper halfplane. After this, we can immediately change to the t-representation in equation (2.24), and we obtain the quasilinear equation for a weakly turbulent plasma:

$$\frac{\partial \Phi(t, \mathbf{v})}{\partial t} = i \frac{e^2}{m^2} \int \frac{d\mathbf{q}}{\mathbf{q}^2} |\mathbf{E}\,{}^0|^2 \mathbf{q} \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega_{\mathbf{q}}{}^0 - \mathbf{q}\mathbf{v} + i0} \mathbf{q} \frac{\partial \Phi(t, \mathbf{v})}{\partial \mathbf{v}} = \pi \frac{e^2}{m^2} \int \frac{d\mathbf{q}}{\mathbf{q}^2} |\mathbf{E}_{\mathbf{q}}{}^0|^2 \mathbf{q} \frac{\partial}{\partial \mathbf{v}} \delta(\omega_{\mathbf{q}}{}^0 - \mathbf{q}\mathbf{v}) \mathbf{q} \frac{\partial \Phi(t, \mathbf{v})}{\partial \mathbf{v}}. \quad (2.26)$$

Let us clarify the meaning of the condition (2.25). Since the right part of (2.25) will be minimal when $\mathbf{v}$ lies in the resonance region of velocities, we can set $\nu \sim \omega_{\mathbf{q}}/q$. For purposes of simplicity, examining the case of Langmuir operations for which $\omega_{\mathbf{q}} \sim \omega_0$, and assuming that $\nu = \omega_0/q_0$, where q_0 is the mean wave number, instead of (2.25) we obtain /522

$$\tau^{-1} \ll \omega_0 \Delta q / q_0 \quad (2.27)$$

(Δq is the width of the wave packet). The quantity τ represents the characteristic relaxation time of the distribution function in the resonance region of velocities, and is determined by the condition

$$\tau \sim D/(\Delta v)^2, \quad D \sim e^2 q^2 \varphi^2 / m^2 \Delta q v, \quad \Delta v = \Delta(\omega_q/q) \sim \omega_0 \Delta q / q^2, \quad (2.28)$$

where D is the quasilinear diffusion coefficient in velocity space, and ϕ is the mean electric field potential of the wave:

$$|\varphi|^2 = \int |\varphi_{\mathbf{q}}|^2 d\mathbf{q} \sim |\varphi_{\mathbf{q}_0}|^2 \Delta q.$$

Substituting (2.28) in (2.27), we find that the condition under which equation (2.24) changes into a quasilinear equation has the form

$$\Delta v = \Delta(\omega_{\mathbf{q}}/q) \gg (e\varphi/m)^{1/2}, \tag{2.29}$$

which coincides with the applicability condition of the quasilinear equation obtained by Vedenov (Ref. 14) from other considerations.

The physical meaning of (2.29) is that the scatter width of the wave phase velocities considerably exceeds the particle oscillation velocity in the potential well of the wave field with the amplitude ϕ. On the other hand, (2.29) can be regarded as the plasma turbulence criterion. Equation (2.26) does not contain terms which describe the "adiabatic" change of the distribution functions in the nonresonance region. In order to obtain these terms, let us turn to the integral in the right part of (2.23), and let us determine the effects of a weak change in the field amplitude. Since $\mathbf{E}_{\mathbf{q}}(\omega)$ becomes particularly large for $\omega = \omega_{\mathbf{q}}{}^0$, the regions where $\omega'' \sim \omega_{\mathbf{q}}{}^0$, $\omega' \sim \omega^0_{-\mathbf{q}} = -\omega_{\mathbf{q}}{}^0$ make the main contribution to the integral in (2.23). Therefore, let us represent the denominator of the integrand in the form $\omega-\omega'-\omega_{\mathbf{q}}{}^0+\omega_{\mathbf{q}}{}^0-\mathbf{qv}$, and let us expand $(\omega-\mathbf{qv}-\omega')$ in power series of $\omega-\omega'-\omega_{\mathbf{q}}$ limiting ourselves to the two expansion terms:

$$\frac{1}{\omega-\mathbf{qv}-\omega'} \approx \frac{1}{\omega_{\mathbf{q}}{}^0-\mathbf{qv}+i0} - \frac{\omega-\omega'-\omega_{\mathbf{q}}{}^0}{(\omega_{\mathbf{q}}{}^0-\mathbf{qv}+i0)^2}. \tag{2.30}$$

Substituting this expression in (2.23) and employing the convolution relationship [see footnote (3)], we obtain the following in the t-representation

$$\frac{\partial\Phi(t,\mathbf{v})}{\partial t}=\pi\left(\frac{e}{m}\right)^2\int\frac{d\mathbf{q}}{\mathbf{q}^2}|\mathbf{E}_\mathbf{q}(t)|^2\,\mathbf{q}\frac{\partial}{\partial\mathbf{v}}\delta(\omega_\mathbf{q}{}^0-\mathbf{q}\mathbf{v})\,\mathbf{q}\frac{\partial}{\partial\mathbf{v}}\Phi(t,\mathbf{v})+$$
$$+\frac{1}{2}\left(\frac{e}{m}\right)^2\int\frac{d\mathbf{q}}{\mathbf{q}^2}\mathbf{q}\frac{\partial}{\partial\mathbf{v}}\left\{\frac{1}{(\omega_\mathbf{q}{}^0-\mathbf{q}\mathbf{v})^2}\left[\frac{d|\mathbf{E}_\mathbf{q}(t)|^2}{dt}\mathbf{q}\frac{\partial}{\partial\mathbf{v}}\Phi(t,\mathbf{v})+2|\mathbf{E}_\mathbf{q}(t)|^2\times\right.\right. \quad (2.31)$$
$$\left.\left.\times\,\mathbf{q}\frac{\partial}{\partial\mathbf{v}}\frac{\partial}{\partial t}\Phi(t,\mathbf{v})\right]\right\},$$

where the principal value of the second integral is chosen. We can apparently disregard the term with $\partial\Phi/\partial t$ in the right part. After this, (2.31) will fully coincide with the quasilinear equation which allows for adiabatic change of the distribution function in the non-resonance region. The /523 applicability conditions of this equation are determined, just as previously, by relationship (2.23).

In a similar way, we can show [for greater details, see (Ref. 19), pages 14, 15] that the equation for the field (2.21) changes into the equation of the quasilinear theory for a weakly turbulent plasma when condition (2.29) is fulfilled:

$$\frac{d\mathbf{E}_\mathbf{k}(t)}{dt}=[-i\omega_\mathbf{k}{}^0+\gamma_k(t)]\,\mathbf{E}_\mathbf{k}(t), \qquad (2.32)$$

$$\gamma_\mathbf{k}(t)=\pi\frac{\omega_0{}^2}{\varepsilon_\mathbf{k}{}'k^2}\frac{\partial\Phi(t,v)}{\partial v}\bigg|_{v=\omega_\mathbf{k}{}^0/k}; \qquad (2.33)$$

where $\omega_K{}^0$ is the real part of the frequency in the linear approximation.

3. QUASILINEAR THEORY OF A "MONOCHROMATIC" WAVE

In this section, we shall examine in detail the use of generalized quasilinear equations (2.21) and (2.23) in studying the development of a nonlinear "monochromatic" wave. The sum in (2.23) now pertains to the discrete selection of the wave vectors $\mathbf{q}$ = nK, n = ±1,2,.... where $2\pi/k$ is the wavelength. The amplitudes of the multiple harmonics will be of a higher order of smallness, as compared with the amplitude of the first harmonics, so that they can be disregarded. Also disregarding the dependence of the wave amplitude on time[6] in equation (2.23), we arrive at equation (2.24), in which the sum consists of two components corresponding to **q** = ±K. Introducing the following notation which is more suitable for this case,

$$\alpha^2 = 2^{1/2} keE_k^0 / m, \qquad u = v - \omega_k / k, \qquad x = ku / \alpha \tag{3.1}$$

(α is on the order of the isolation frequency of particles trapped by the potential well of the wave; correspondingly, α/k is on the order of the trapped particle velocity), we obtain the principal equation in the following form

$$\Phi(\omega, x) = \frac{i}{\omega} f(x) - \alpha^2 \frac{\partial}{\partial x} \frac{1}{\omega^2 - \alpha^2 x^2} \frac{\partial}{\partial x} \Phi(\omega, x). \tag{3.2}$$

In order to solve this equation, let us introduce the function

$$\Psi(\omega, x) = \frac{\alpha^2}{\omega^2 - \alpha^2 x^2} \frac{\partial}{\partial x} \Phi(\omega, x). \tag{3.3}$$

Differentiating both parts of equation (3.2) with respect to x and substituting (3.3), we obtain

$$\frac{\partial^2}{\partial x^2} \Psi(\omega, x) + \left(\frac{\omega^2}{\alpha^2} - x^2\right) \Psi(\omega, x) = \frac{i}{\omega} \frac{df}{dx}. \tag{3.4}$$

The solution of equation (3.4) can be represented in the form of an expansion with respect to normalized parabolic cylinder functions $\psi_n(x)$:

$$\psi_n(x) = (2^n n! \, \pi^{1/2})^{-1/2} e^{-x^2/2} H_n(x), \qquad \int_{-\infty}^{\infty} \psi_n^2(x) \, dx = 1,$$

where $H_n(x)$ are Hermite polynomials. The $\psi_n(x)$ functions satisfy the equation

$$\frac{d^2\psi_n(x)}{dx^2} + (2n + 1 - x^2) \psi_n(x) = 0. \tag{3.5}$$

Assuming that /524

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(6) As will be shown below, this is valid when condition (3.28) is fulfilled.

$$\frac{df}{dx}=\sum_{n=0}^{\infty}\beta_n\psi_n(x), \qquad \beta_n=\int_{-\infty}^{\infty}\psi_n(x)\frac{df}{dx}dx, \tag{3.6}$$

from (3.3) and (3.4) we obtain

$$\begin{aligned}\frac{\partial\Phi(\omega,x)}{\partial x}&=i\frac{\omega^2-\alpha^2x^2}{\omega}\sum_n\frac{\beta_n\psi_n(x)}{\omega^2-(2n+1)\alpha^2}=\\&=\frac{i}{\omega}\frac{df}{dx}+\frac{i\alpha^2}{\omega}\sum\frac{2n+1-x^2}{\omega^2-(2n+1)\alpha^2}\beta_n\psi_n(x).\end{aligned} \tag{3.7}$$

Changing to the t-representation, we obtain

$$\frac{\partial\Phi(t,x)}{\partial x}=\frac{df}{dx}-\sum_n\frac{2n+1-x^2}{2n+1}\beta_n\psi_n(x)[1-\cos\alpha t\sqrt{2n+1}]. \tag{3.8}$$

In order to calculate the distribution function $\Phi(t,x)$, let us replace $(x^2-2n-1)\psi_n(x)$ in (3.8) by $d^2\psi_n(x)/dx^2$, after which elementary integration is performed. Returning to the variable u, according to (3.1) we obtain

$$\Phi(t,u)=f(u)+\frac{\alpha}{k}\sum\frac{\beta_n}{2n+1}\frac{d}{du}\psi_n\left(\frac{ku}{\alpha}\right)(1-\cos\alpha t\sqrt{2n+1}). \tag{3.9}$$

It can be seen from (3.9) that $\Phi(t,u)$ differs very little from its initial value $f(u)$ for $t \ll \alpha^{-1} \sim (keE_0/m)^{-1/2}$, where E_0 is the wave amplitude. In the case of $t > \alpha^{-1}$, the second term in (3.9) oscillates with a change in t, and it rapidly decreases in the case of

$$u=v-\omega_k/k\gg\alpha/k\sim(eE_0/km)^{1/2},$$

i.e., the quasilinear distribution function is deformed only close to the resonance velocity $v = \omega_k/k$ in the velocity interval on the order of the oscillation velocity of particles "trapped" by a wave. The oscillation frequency of the distribution function close to phase velocity, as can be seen from (3.9), is on the order of α, which coincides with the particle oscillation frequency in the potential well of a wave.

Now substituting (3.7) in (2.22) for $\varepsilon_k(\omega,\omega')$, we obtain

$$\varepsilon_k(\omega, \omega') = \frac{i}{\omega - \omega'}\Big[1 + \frac{\omega_0^2}{k^2}\int \frac{dv}{\omega - kv} k \frac{df}{dv} + \\ + \frac{\omega_0^2}{\alpha}\sum_n \frac{\beta_n}{(\omega-\omega')^2 - (2n+1)\alpha^2}\int du \frac{(2n+1)\alpha^2 - k^2u^2}{\omega - \omega_k - ku}\psi_n\Big(\frac{ku}{\alpha}\Big)\Big]. \tag{3.10}$$

Utilizing (3.10) and (2.21) and performing elementary transformations, we obtain

$$E_k(\omega) = \frac{4\pi e N}{k\varepsilon_k(\omega)}\int \frac{g_k(v)}{\omega - kv}dv - \frac{i\omega_0^2}{2\pi\alpha\varepsilon_k(\omega)}\sum_n \beta_n \times \\ \times \int du \psi_n\Big(\frac{ku}{\alpha}\Big)\frac{(2n+1)\alpha^2 - k^2u^2}{\omega - \omega_k - ku}\int \frac{d\omega' E_k(\omega')}{(\omega - \omega')[(\omega - \omega')^2 - (2n+1)\alpha^2]}, \tag{3.11}$$

where $\varepsilon_k(\omega)$ is determined by formula (2.11). Multiplying both parts of (3.11) by $i(\omega - \omega_k)$ [ω_k - the oscillation frequency in the linear approximation, see (2.14)] and changing to the t-representation, we obtain

$$\frac{dE_k(t)}{dt} = -i\omega_k E_k(t) - \frac{\omega_0^2}{\varepsilon_k'\alpha^3}\sum_n \frac{\beta_n}{2n+1}\int du\psi_n\Big(\frac{ku}{\alpha}\Big)\times \\ \times [(2n+1)\alpha^2 - k^2u^2]\int_0^t dt'(1 - \cos\alpha t'\sqrt{2n+1})\times \\ \times E_k(t')\exp[i(ku + \omega_k)(t' - t)]. \tag{3.12}$$

In obtaining (3.12), we have disregarded all of the properties of the right part of (3.11), which are located below the point ω_k, since they make an exponentially small contribution in the case of rather large t. In addition, we have omitted the term $\int d\nu g_k(\nu) e^{-ik\nu t}$, which rapidly decreases with an increase in t, due to the factor $e^{-ik\nu t}$ which oscillates for large t. The characteristic time of this decrease is $(k\nu g)^{-1}$, where ν_g is the effective "width" of the function $g_k(\nu)$. We have assumed that this time is small as compared with other characteristic times which determine the change in the field amplitude. Utilizing (3.5) and the following relationship [see (Ref. 20)] /525

$$\int_{-\infty}^{\infty} \psi_n(y) e^{iyz}\, dy = (2\pi)^{1/2} i^n \psi_n(z), \tag{3.13}$$

We can perform integration with respect to u in the second term of the right part of (3.12). As a result, we obtain

$$\frac{dE_k(t)}{dt} = -i\omega_k E_k(t) - (2\pi)^{1/2} \frac{\omega_0^2 \alpha^2}{\varepsilon_k' k} \sum_n \frac{i^n \beta_n}{2n+1} \int_0^t dt' (t'-t)^2 \times$$
$$\times\, \psi_n[\alpha(t'-t)] (1 - \cos \alpha t' \sqrt{2n+1}) E_k(t') \exp [i\omega_k (t'-t)]. \tag{3.14}$$

The integrand in (3.14) contains the product of the following expression, which changes rapidly with a change in t',

$$\psi_n[\alpha(t'-t)] (1 - \cos \alpha t' \sqrt{2n+1})$$

(the characteristic time for the change is $\tau \lesssim \alpha^{-1}$) by the quantity $E_k(t') \exp (i\omega_k t')$, which equals the slowly-changing field amplitude $E_k^0(t')$. Disregarding the change $E_k^0(t')$ during the time α^{-1}, we can take $E_k^0(t')$ out from under the integral sign. Performing simple transformations after this, we obtain

$$\frac{dE_k(t)}{dt} = [-i\omega_k + \theta_k(t)] E_k(t), \tag{3.15}$$

$$\theta_k(t) = -(2\pi)^{1/2} \frac{\omega_0^2}{\varepsilon_k' k\alpha} \sum_n (-1)^n \int_0^{\alpha t} d\tau \{\beta_{2n} [\psi_{2n}(\tau) - \psi_{2n}(0) \cos \tau \sqrt{4n+1}] -$$
$$- i\beta_{2n+1} [\psi_{2n+1}(\tau) - (4n+3)^{-1/2} \psi'_{2n+1}(0) \sin \tau \sqrt{4n+3}]\}. \tag{3.16}$$

Integrating the equation, we will have

$$E_k(t) = E_k(0) \exp \left\{ -it \left[\omega_k^0 + i\gamma_k + it^{-1} \int_0^t dt' \theta_k(t') \right] \right\}. \tag{3.17}$$

It can be seen from (3.17) that $t^{-1} \mathrm{Re} \int_0^t dt' \theta_k(t')$ represents a change /526 in the increment, and $t^{-1} \mathrm{Im} \int_0^t dt' \theta_k(t')$ represents the real part of the frequency caused by a distortion of the distribution function due to the reverse effect of a wave. After studying (3.16), one can readily state that it converges fairly rapidly, so that the order of magnitude of $\theta_k(t)$ does not exceed the order of the first term in the series. Calculating the quantity β_n from (3.6) under the assumption that $f(v)$ is the Maxwell distribution, we readily find that

$$t^{-1}\int_0^t \theta_k(t')\,dt' \lesssim \gamma_k, \tag{3.18}$$

where γ_k is the linear theory increment. Since $\gamma_k \ll \omega_k{}^0$, the nonlinear correction to the real part of the frequency can be disregarded. On the other hand, the correction to the linear increment is very considerable. Therefore, let us examine the following quantity in greater detail

$$\Gamma_k(t) = \gamma_k + t^{-1}\,\mathrm{Re}\int_0^t \theta_k(t')\,dt', \tag{3.19}$$

which represents an increment which is dependent on time.

Let us first study $\Gamma_k(t)$ for small $t(\alpha t \ll 1)$. We shall make use of the fact that, for rather small arguments and large orders, the functions of a parabolic cylinder have the following asymptotic representation (Ref. 20):

$$\psi_{2n}(z) = \psi_{2n}(0)\{\cos z\sqrt{4n+1} + O[z^{5/2}(4n+1)^{-1/4}]\}. \tag{3.20}$$

Taking (3.20) into account, we find that

$$t^{-1}\int_0^t dt' \int_0^{\alpha t'} d\tau\,[\psi_{2n}(\tau) - \psi_{2n}(0)\cos\tau\sqrt{4n+1}] < \psi_{2n}(0)\,(\alpha t)^{7/2}(4n+1)^{-1/4}, \tag{3.21}$$

and it can thus be seen that for $\alpha t \ll 1$ the difference between $\Gamma_k(t)$ and the linear increment is a small quantity on the order of $(\alpha t)^{7/2}$. We should note that for $\alpha t \sim 1$ the left part of (3.21) decreases rapidly with an increase in n, due to which fact we can confine ourselves to a few of the first terms in the series in expression (3.19) for $\Gamma_k(t)$ in the case of $t \sim \alpha^{-1}$. Since these terms are on the order of the linear increment

γ_k, $\Gamma_k(t)$ differs significantly from γ_k in the case of

$$t \sim \alpha^{-1} \sim (keE_k^0/m)^{1/2}.$$

Let us now study $\Gamma_k(t)$ for $t \gtrsim \alpha^{-1}$. Let us rewrite formula (3.19) in the following form

$$\Gamma_k(t) = \gamma_k - (2\pi)^{1/2} t^{-1} \frac{\omega_0^2}{\varepsilon_k' k\alpha} \int_0^t dt' \left\{ \sum_n (-1)^n \beta_{2n} \int_0^\infty d\tau \psi_{2n}(\tau) - \right.$$
$$\left. - \sum_n (-1)^n \beta_{2n} \int_{\alpha t'}^\infty d\tau \psi_{2n}(\tau) \right\} + (2\pi)^{1/2} t^{-1} \frac{\omega_0^2}{\varepsilon_k' k\alpha^2} \times \tag{3.22}$$
$$\times \sum_n \frac{(-1)^n \beta_{2n}}{4n+1} \psi_{2n}(0) \left(1 - \cos \alpha t \sqrt{4n+1}\right).$$

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Utilizing (3.13) and (3.6), we can rewrite the first term in the braces in the following way:

$$\sum_n (-1)^n \beta_{2n} \int_0^\infty \psi_{2n}(\tau) d\tau = \left(\frac{\pi}{2}\right)^{1/2} \sum \beta_{2n} \psi_{2n}(0) =$$
$$= \left(\frac{\pi}{2}\right)^{1/2} \frac{k}{\alpha} \frac{df}{dv} \Big|_{v=\omega_k/k}. \tag{3.23}$$

The substitution of (3.23) in (3.22) leads to exact compensation for the first term γ_k in (3.22). We can represent the second sum in the braces (3.22) in the following form

$$\sum_n (-1)^n \beta_{2n} \int_{\alpha t'}^\infty d\tau \psi_{2n}(\tau) = (2\pi)^{-1/2} \sum \beta_{2n} \int_{\alpha t'}^\infty d\tau \int_{-\infty}^\infty dx \psi_{2n}(x) e^{itx} =$$
$$= \alpha (2\pi)^{-1/2} \int_{t'}^\infty dt'' \int_{-\infty}^\infty du \frac{df}{du} \cos(kut''). \tag{3.24}$$

This expression vanishes for $t \to \infty$ (taking the fact into account that $u = v - \omega_k/k$, and $f(v) \sim \exp(-v^2/v_T^2)$, and it can be readily seen that the term (3.24) rapidly begins to decrease for $t > (kv_T)^{-1}$. Thus, for rather large t, the quantity $\Gamma_k(t)$ assumes the following form

$$\Gamma_k(t) = \frac{2(2\pi)^{1/2}\omega_0^2}{\varepsilon_k' k\alpha^2 t} \sum_n \frac{(-1)^n \beta_{2n}}{4n+1} \psi_{2n}(0) \sin^2 \alpha t\sqrt{n+1/4}. \tag{3.25}$$

Thus, the nonlinear increment $\Gamma_k(t)$, for times which are small are compared with the particle oscillation period in the potential well of the wave ($\alpha t \ll 1$), is close to the linear increment γ_k. For large t, the quantity $\Gamma_k(t)$, oscillating, attenuates as t^{-1}. However, the wave amplitude, determined by the expression

$$E_k^0(t) = E_k^0(0) \exp[t\Gamma_k(t)], \tag{3.26}$$

does not strive to a stationary value in the case of $t \to \infty$, since oscillations of the quantity $t\Gamma_k(t)$ are not attenuated. In order of magnitude, the characteristic period of these oscillations is α^{-1} - the particle oscillation period in the potential well of a wave. The oscillation amplitude of the quantity $t\Gamma_k(t)$ is on the order of

$$\overline{t\Gamma_k(t)} = \frac{(2\pi)^{1/2}\omega_0^2}{\varepsilon_k' k\alpha^2} \sum (-1)^n \frac{\beta_{2n}\psi_{2n}(0)}{4n+1} \sim \frac{\gamma_k}{\alpha}. \tag{3.27}$$

It was constantly assumed above that the field amplitude changes very little during the time of the wave nonlinear development. It follows from (3.26) and (3.27) that this holds under the condition that

$$\gamma_k / \alpha \ll 1. \tag{3.28}$$

Under this condition, the exponent in (3.26) can be expanded in series, /528 and the average wave amplitude is

$$\overline{E_k^0(t)} \approx E_k^0(0) \left[1 + \frac{(2\pi)^{1/2}\omega_0^2}{\varepsilon_k' k\alpha} \sum_n \frac{(-1)^n \beta_{2n}}{4n+1} \psi_{2n}(0)\right]. \tag{3.29}$$

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